II. $p = 2 \sin 36° = 4s\sqrt{1 - s^2}$, where $s = \sin 18° = \cos 72°$. But 72° is four times 18°; therefore s is a root of the equation

$$8s^4 - 8s^2 - s + 1 = 0.$$

The factors $s - 1$ and $2s + 1$ correspond to the angles 90° and $-30°$. Thus we have $4s^2 + 2s - 1 = 0$, and the positive root is $s = \frac{1}{4}(\sqrt{5} - 1)$, from which we get p and a.

III. Let d be the side of the regular decagon inscribed in the same circle; then d being a chord of an arc in a unit circle and p that of twice the same arc,

$$d^2 = 2 - \sqrt{4 - p^2}.$$

Let O be the center of the circle, AB and AC chords equal to p and d, F the foot of the perpendicular from A upon OB, and G the middle point of AB. The four points O, F, G and A are concyclic and $BA \cdot BG = BO \cdot BF$.

Now the triangle AOF is congruent to the right triangle formed by the radius and apothem of the decagon, having the same angles and hypothenuse equal to 1. Therefore $OF = d/2$ and our equation may be written

$$p \cdot \frac{p}{2} = 1\left(1 - \frac{d}{2}\right) \text{ or } p^2 = 2 - d.$$

Eliminating d and removing the factor $p^2 - 3$, we have

$$p^4 - 5p^2 + 5 = 0$$

or, since p is only a very little more than 1, $p^2 = \frac{1}{2}(5 - \sqrt{5})$. This gives p and then a as above.

Also solved by MICHAEL GOLDBERG, R. M. MARSHALL, J. Q. McNATT, ARTHUR PELLETIER, A. V. RICHARDSON, and L. S. SHIVELY. Some have interpreted the problem as calling for a geometrical construction rather than an algebraic solution.

2859 [1920, 428]. Proposed by L. S. DEDERICK, U. S. Naval Academy.

Derive an expression for the limit of error in evaluating a definite integral by Simpson's Rule.

I. SOLUTION BY A. A. BENNETT, University of Texas.

Let the desired definite integral be $\int_{a-h}^{a+h} f(x)dx$, where $h > 0$. If the function $f(x)$ be supposed capable of a Taylor expansion through the term in $(x - a)^3$, with a remainder, a simple solution for the problem is obtained by employing this expansion, the form of the result depending upon the form of the remainder adopted.

The problem as stated should not depend upon the existence of derivatives at any point of the interval. A solution not involving such derivatives is the following:

$$\left| \int_{a-h}^{a+h} f(x)dx - \frac{h}{3}\{f(a+h) + 4f(a) + f(a-h)\}\right| \leqq \text{Max}\{h|x-a|^2|D(x) - D(a+h)|\}, \quad (1)$$

where $(x - a)^2 D(x) \equiv f(x) - 2f(a) + f(2a - x)$.

If $f(x)$ has a second derivative at $x = a$, this is of course the value approached by $D(x)$ as x approaches a. To prove (1), write

$$g(x) \equiv h(x - a)^2\{D(x) - D(a + h)\}.$$

Expanding and integrating we have

$$\int_{a-h}^{a+h} g(x)dx = 2h\left[\int_{a-h}^{a+h} f(x)dx - \frac{h}{3}\{f(a+h) + 4f(a) + f(a-h)\}\right].$$

Since $\left|\int_{a-h}^{a+h} g(x)dx\right| \leqq 2h \text{ Max } |g(x)|$, (1) is proved.

It is to be noted that Max $\{h|x - a|^2|D(x) - D(a+h)|\} = 0$, for polynomials of less than the fourth order, so that the formula given by Simpson's rule is exact in these cases.

II. Note on Professor Bennett's solution by H. P. Manning, Providence, R. I., and Otto Dunkel, Washington University.

If $f(x)$ can be expanded to four terms with the Lagrange remainder, the substitution of this expansion will show that max $|g(x)| \leqq \frac{h^5}{3!}$ max $|f^{iv}(x)|$, but substitution in the integral $\int_{a-h}^{a+h} g(x)dx$ gives for the error the smaller limit $\frac{2}{45} h^5$ max $|f^{iv}(x)|$, the same that is given by the substitution for $f(x)$, and for $f(a+h)$ and $f(a-h)$ in Simpson's formula itself. This limit is given in E. B. Wilson's *Advanced Calculus*, 1912, pages 76–77, exercises 23 and 24. However, a limit still smaller, namely, $\frac{h^5}{90}$ max $|f^{iv}(x)|$, has been found and is given by P. J. Daniell in this Monthly, *1917*, 110, and also by C. J. de la Vallée-Poussin, *Cours d'Analyse Infinitésimale*, volume 1, third edition, 1914, page 396. The method employed by the latter can be applied to the integral $\int_{a-h}^{a+h} g(x)dx$ and leads to the same result. Thus Professor Bennett's expression, obtained without assuming any expansion, leads to the results already found for functions capable of expansion to four terms and a remainder.

In 1874 Chevilliet (*Comptes Rendus de l'Académie des Sciences*, volume 78, page 1841), by taking the infinite expansion of $f(x)$, shows that the first term of the error is $-\frac{h^5}{90} f^{iv}(a)$, and when h is sufficiently small this approximates to the limit given by de la Vallée-Poussin. This result is obtained in the same way in Kiepert's *Grundriss der Differential- und Integralrechnung*, Teil 2, seventh edition, 1900, pages 335–336.

In Heine's *Handbuch der Kugelfunktionen*, Band 2, Theil 1, 1881, there is an exhaustive discussion of mechanical quadrature, and expressions are obtained for the errors in the Newton-Cotes method and in the method of Gauss. In particular, the fraction $-1/90$ can be obtained by multiplying $1/4!$ by the $-4/15$ given in the table on page 9.

William Hoover gave the reference to Wilson's *Calculus*, and H. E. Jordan to Kiepert's work.

2868 [1920, 482]. Proposed by H. S. Uhler, Yale University.

Let the evolute of a given curve be called the evolute of the first order, let the evolute of the first evolute be called the evolute of the second order, etc. Then, being given the following parametric equations in which a is a constant and γ is the parameter, namely,

$$x = (1 + 2\sin^2 \gamma)\cos \gamma - a \sin 2\gamma, \qquad y = 2\sin^3\gamma + a \cos 2\gamma,$$

find: (a) the parametric equations of the evolute of order n, both for n even and for n odd;
(b) a formula for the total length of the nth evolute;
(c) a formula for the total area of the nth evolute;
(d) the sum of the lengths of all the evolutes from $n = 1$ to $n = \infty$; and
(e) the sum of the areas of all the evolutes from $n = 1$ to $n = \infty$.

Note. The original equations represent the envelope required in problem 2819 (*1920*, 134).

I. Solution by F. L. Wilmer, Omaha, Neb., and H. P. Manning, Providence, R. I.

One may note that the equation of the normal to the given curve can be put into the p-form with p a simple function of the parameter. Then differentiation with respect to the latter must give the equation of a perpendicular meeting this line in the corresponding point of the evolute, and so normal to the latter. In this way are obtained the parametric equations of the evolute, and by repetition those of the nth evolute.

We can write the given equations

$$x = 3\cos \gamma - 2\cos^3 \gamma - a \sin 2\gamma,$$
$$y = \qquad\qquad 2\sin^3 \gamma + a \cos 2\gamma;$$

and from these it follows that $dy/dx = \tan 2\gamma$.